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Irreducible Representations of a Parafield and the Connection of the Parafield with Usual Fields

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Abstract

All irreducible separable representations of the non-relativistic para-Fermi field of order 3 in the configuration space are considered. The existence of many different irreducible representations of the parafield permits us to interpret the excited states of this field as the states of particles with internal degrees of freedom. These states can be labelled by the Young patterns and the eigenvalues of internal quantum number like baryonic and hypercharges. The parafield theory is shown to be equivalent to the theory of three kinds of ordinary fermions, like quarks, and one of them, 'strange', can be distinguished from the other two by means of its interaction, not only statistically but also dynamically. Thus the parafield theory is shown to be equivalent to some model of the physical $SU(3)$ symmetry of hadrons when the strong and medium-strong interactions could be switched on but the electromagnetic and weak interactions should be switched off.

1. *Introduction*

We shall be examining a non-relativistic Schrödinger spinor field $\psi(\mathbf{x},\sigma,t)$, in which $\mathbf{x}$, σ and t mean the space, spin and time variables, respectively. In what follows we shall omit the dependence of functions on spin variables and assume that the variable $\mathbf{x}$ includes also the spin variable and the integration implies a summation over the spin variables. Below, the time variable is omitted too.

We shall quantize the field following Green (1953), in accordance with the Green trilinear commutation relations

$$[X,[Y,Z]] = 2\{X, Y\}Z - 2\{X,Z\}\, Y \tag{1.1}$$

Here we used a compact symbolic form for all the relations which are obtained by the substitutions of the creation operator, $\psi^*(\mathbf{x})$, or the annihilation operator, $\psi(\mathbf{x})$, for any of the symbols X, Y, Z. There is a double-commutator in the left-hand side of equation (1.1). The braces in the

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right-hand side of equation (1.1) mean the Volkov symbols (Volkov, 1959, 1960) that are symmetrical and take the numerical values

$$\left.\begin{aligned}\{\psi(\mathbf{x}),\psi^*(\mathbf{y})\} = \{\psi^*(\mathbf{y}),\psi(\mathbf{x})\} = \delta^{(3)}(\mathbf{x}-\mathbf{y})\\ \{\psi(\mathbf{x}),\psi(\mathbf{y})\} = \{\psi^*(\mathbf{x}),\psi^*(\mathbf{y})\} = 0\end{aligned}\right\} \tag{1.2}$$

where $\delta^{(3)}(\mathbf{x})$ is the three-dimensional Dirac δ-function.

Using equations (1.1), the existence of the vacuum state, for which

$$\psi(\mathbf{x})|0\rangle = 0 \tag{1.3}$$

is identically valid, and the positiveness of the state vector norms (Greenberg & Messiah, 1965) have shown that there is also

$$\psi(\mathbf{x})\psi^*(\mathbf{y})|0\rangle = s\delta^{(3)}(\mathbf{x}-\mathbf{y})|0\rangle \tag{1.4}$$

where s is a non-negative integer 0, 1, 2, etc. This number defines the maximum number of particles in the symmetrical state. For this reason the theory is called the para-Fermi statistics of order s and is labelled by pF_s-statistics. Respectively, the field obeying equations (1.1) is called a parafield. It can be easily proved that the usual Fermi field satisfies equations (1.1) identically and that the Fermi statistics corresponds to the case $s = 1$.

Only the Fock representation, i.e. the representation with a unique vacuum state, is usually considered in literature devoted to the investigation of the parafield. It is considered as a unique representation convenient for the field theory. In reality, there are many different irreducible (separable) representations of equations (1.1) for para-Fermi statistics of a given order from a mathematical point of view (Ryan & Sudarshan, 1963). The aim of the present paper is the investigation and the imparting of the definite physical meaning to the states of these new irreducible representations for the case of para-Fermi statistics of order 3 (pF_3). Of course, if we investigate all irreducible representations of a parafield we do the same for the unique Fock representation too. An analogous consideration was performed in the previous paper (Govorkov, 1969) for the case of para-Fermi statistics of order 2 (pF_2).

Like the Fock representation, all the new irreducible representations of equations (1.1) contain the states which satisfy equation (1.3). It will be shown below that these states will not correspond to the vacuum states but will correspond to one-, two-, etc., particle states.

The existence of many different representations of equations (1.1) for one parafield permits us to interpret its excited states as the states of particles with internal degree of freedom.

The case of pF_2-statistics was investigated in the previous paper (Govorkov, 1969) and it was shown that it is equivalent to the usual Fermi statistics of particles with two-valued internal degree of freedom like isospin. In the present paper we investigate the case of pF_3-statistics and show its equivalence to the usual Fermi statistics of particles with internal degree of freedom like isospin and strangeness. At the same time, we establish the restricted meaning of a connection between the parafield theory and the possible $SU(3)$ theory of usual fermions.

In Section 2 our aim is to build all irreducible representations of equations (1.1) in the configuration space for the case of pF_3-statistics. It will be convenient to use the so-called Green-Ansatz that is some reducible representation of equations (1.1). We build irreducible representations of equations (1.1) extracting them from its irreducible space. It should be taken into account that the Green-Ansatz is a very convenient but unnecessary tool for our aims and can be replaced by a pure algebraic construction, as will be shown below.

In Section 3 we define the class of many-particle functions on which pF_3-statistics is realized. These functions should be symmetrized along Young patterns. Then the symmetrized functions determine the probability of finding the paraparticles in their definite symmetrized states.

In Section 4 we find a unitary symmetry of paraparticle states, in completing it we again use the Green-Ansatz. We compare the parafield states with the states of usual Fermi field with three-valued internal degree of freedom in the framework of the Green-Ansatz. Then we return our consideration to the proper parafield theory and, in each, its irreducible representation. We determine an operator of hypercharge in the parafield framework. Each state from the irreducible representations can be labelled by the Young pattern and the eigenvalue of an internal quantum number like hypercharge (strangeness). Then we discuss some parafield interaction which breaks up the unitary symmetry of parafield states.

Remark. The inferences of Sections 2–4 allow us to resolve the question about the connection between the first-quantized theory of many paraparticles and the second-quantized parafield theory. This question was raised in literature many times (Kamefuchi & Takahashi, 1962; Galindo & Indurain, 1963; Greenberg, 1966; Landshoff & Stapp, 1967; Yamada, 1968; Carpenter, 1970; Ohnuki & Kamefuchi, 1970; Stolt & Taylor, 1970).

In conclusion we discuss our results and compare them with the results of other authors.

2. *Irreducible Representations of the para-Fermi Field*

Green found in his original article (Green, 1953) a simple solution of equations (1.1) which is called the Green-Ansatz. It consists of the following. Consider an s number of different mutually commuting Fermi fields

$$\left.\begin{aligned} \psi^A(\mathbf{x})\psi^{B*}(\mathbf{y}) + (2\delta_{AB}-1)\psi^{B*}(\mathbf{y})\psi^A(\mathbf{x}) &= \delta_{AB}\delta^{(3)}(\mathbf{x}-\mathbf{y}) \\ \psi^A(\mathbf{x})\psi^B(\mathbf{y}) + (2\delta_{AB}-1)\psi^B(\mathbf{y})\psi^A(\mathbf{x}) &= 0 \\ \psi^{A*}(\mathbf{x})\psi^{B*}(\mathbf{y}) + (2\delta_{AB}-1)\psi^{B*}(\mathbf{y})\psi^{A*}(\mathbf{x}) &= 0 \end{aligned}\right\} \tag{2.1}$$

where δ_{AB} is the Kronecker symbol and indices A and B take a value from the set 1, 2, ..., s. The solution of equations (1.1) is

$$\psi(\mathbf{x}) = \sum_{A=1}^{s} \psi^A(\mathbf{x}), \qquad \psi^*(\mathbf{x}) = \sum_{A=1}^{s} \psi^{A*}(\mathbf{x}) \tag{2.2}$$

The sums (2.2) satisfy also equations (1.3) and (1.4) and

$$\boxed{}\boxed{}\boxed{\cdots}\boxed{}\quad \psi(\mathbf{x}_1)\,\psi(\mathbf{x}_2)\ldots\psi(\mathbf{x}_{s+1})=0 \tag{2.3}$$

where the Young pattern $\boxed{}\boxed{}\boxed{\cdots}\boxed{}$ implies the symmetric combination in arguments $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{s+1}$. Thus the sums (2.2) really correspond to the para-Fermi statistics of order s.

We label the space of the Green-Ansatz, i.e. the Fock irreducible representation of equations (2.1), by $\mathscr{A}$. However, this space gives us the reducible representation of equations (1.1). Now our immediate task is to single the irreducible representations of equations (1.1) out from this large space $\mathscr{A}$.

To this end we consider in the space $\mathscr{A}$ the vectors

$$|\mathbf{x}_1,\ldots,\mathbf{x}_q\rangle = \sum_{A_1,\ldots,A_q=1}^{s} y_{A_1\ldots A_q}\psi^{A_1*}(\mathbf{x}_1)\ldots\psi^{A_q*}(\mathbf{x}_q)\,|0\rangle \tag{2.4}$$

We choose the coefficients $\mathbf{y}_{A_1\ldots A_q}$ for these vectors so that the latter should satisfy the requirement

$$\psi(\mathbf{x})|\mathbf{x}_1,\ldots,\mathbf{x}_q\rangle = 0 \tag{2.5}$$

We call the vectors (2.4) minor vectors. The vacuum vector $|0\rangle$ is included in the set of minor vectors with $q=0$.

Equation (2.5) imposes definite restrictions on the coefficients $\mathbf{y}_{A_1\ldots A_q}$ of minor vectors (2.4). It can be shown (see Appendix) that the fulfilment of equation (2.5) implies the following properties of minor vectors. They should satisfy the equations

$$\begin{aligned}\psi(\mathbf{x})\,\psi^*(\mathbf{y})\,|\mathbf{x}_1,\ldots,\mathbf{x}_q\rangle = {} & s\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\,|\mathbf{x}_1,\ldots,\mathbf{x}_q\rangle \\ & -2\delta^{(3)}(\mathbf{x}-\mathbf{x}_1)\,|\mathbf{y},\mathbf{x}_2,\ldots\mathbf{x}_q\rangle \\ & -\ldots \\ & -2\delta^{(3)}(\mathbf{x}-\mathbf{x}_q)\,|\mathbf{x}_1,\ldots,\mathbf{x}_{q-1},\mathbf{y}\rangle\end{aligned} \tag{2.6}$$

A minor vector with unique argument should obey

$$\boxed{}\boxed{}\boxed{\cdots}\boxed{}\quad \psi^*(\mathbf{x}_1)\ldots\psi^*(\mathbf{x}_{s-1})\,|\mathbf{x}_s\rangle = 0 \tag{2.7}$$

where the Young pattern indicates the symmetric combination in arguments $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_s$.

The minor vectors are antisymmetrical for the particular cases $s=2$ and $s=3$

$$|\mathbf{x}_1,\ldots,\mathbf{x}_i,\mathbf{x}_{i+1},\ldots,\mathbf{x}_q\rangle = -|\mathbf{x}_1,\ldots,\mathbf{x}_{i+1},\mathbf{x}_i,\ldots,\mathbf{x}_q\rangle \tag{2.8}$$

In accordance with (2.8) the minor vectors are normalized in the following manner

$$\begin{aligned}\langle\mathbf{x}_q',\ldots,\mathbf{x}_2',\mathbf{x}_1'|\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_q\rangle = {} & \sum_{\mathscr{P}}\delta_{\mathscr{P}}\,\delta^{(3)}(\mathbf{x}_1-\mathbf{x}'_{\mathscr{P}_1})\,\delta^{(3)}(\mathbf{x}_2-\mathbf{x}'_{\mathscr{P}_2}) \\ & \times\ldots\delta^{(3)}(\mathbf{x}_q-\mathbf{x}'_{\mathscr{P}q})\end{aligned} \tag{2.9}$$

where the sum is taken over all $q!$ permutations of indices $1, 2, \ldots, q$ and $\delta_{\mathscr{P}}$ is the signature of permutation $\mathscr{P}$. As usual, the signature is equal to $+1$

for an even number of transpositions, i.e. mutual permutations of two indices, and −1 for an odd number of transpositions.

Neither of equations (2.8) and (2.9) is true for the cases $s \geqslant 4$.

Now each irreducible representation of equations (1.1) is obtained by means of action of all possible products of paraparticle creation operators on each minor vector. The general vector of this irreducible representation is

$$|\Psi\rangle = \sum_{p=0}^{\infty} \int d\mathbf{x}_1 \ldots \int d\mathbf{x}_{p+q}\, \Psi^{(p,q)}(\mathbf{x}_1, \ldots, \mathbf{x}_p; \mathbf{x}_{p+1}, \ldots, \mathbf{x}_{p+q})$$
$$\times\, \psi^*(\mathbf{x}_1) \ldots \psi^*(\mathbf{x}_p)|\mathbf{x}_{p+1}, \ldots, \mathbf{x}_{p+q}\rangle \tag{2.10}$$

The arguments in the amplitudes of vectors (2.10) are split by the semicolon into two clusters. The arguments standing before the semicolon are related to the paraparticle creation operators, $\psi^*(\mathbf{x}_i)$, and are called primary arguments. The arguments standing after the semicolon are related to the minor vectors and are called secondary arguments. Each irreducible representation of equations (1.1) is characterized by the fixed number q of the secondary arguments and the variable number of the primary arguments.

From this point of presentation we consider only the third-order para-Fermi statistics. It can be proved that the number of linearly independent minor vectors (2.4), with fixed number of arguments q for this case in the space of the Green-Ansatz, is equal to $q + 1$. All these vectors possess the same characteristics (2.5)–(2.9). Therefore they determine the equivalent irreducible representations of equations (1.1). The minor vectors of equivalent irreducible representations may be taken orthogonal to one another. In their turn the spaces of the irreducible representations constructed on the minor vectors with different numbers of arguments are orthogonal due to equation (2.5). Thus we split the large space of the Green-Ansatz into the orthogonal subspaces of irreducible representations of equation (1.1). We label the space of a given irreducible representation of equations (1.1) by $\mathscr{B}_q$ with number q from the sequence 0, 1, 2, etc. Then we can write our decomposition of large space of Green-Ansatz into the irreducible representations of equations (1.1) in the form of a direct sum

$$\mathscr{A} = \bigoplus_{q \in n} (q+1)\mathscr{B}_q, \qquad n = \{0, 1, 2, \ldots\} \tag{2.11}$$

In this direct sum the multipliers $(q + 1)$ indicate the number of the equivalent irreducible representations.

Thus all irreducible representations of equations (1.1) pick out from the space of the Green-Ansatz. The fact that we found all separable irreducible representations of equations (1.1) by this manner follows from the analogy of the Green algebra with the algebra of a rotational group in the odd-dimensional Euclidean space when the dimensionality goes to infinity (Kamefuchi & Takahashi, 1962; Ryan & Sudarshan, 1963). The Green-Ansatz corresponds in the general case of pF_s-statistics to the Kronecker direct product of the number s of fundamental (spinor) representations of a rotational group.

We now disengage ourselves from the Green-Ansatz. The requirements (2.5)–(2.9) must be postulated now as the existence of the vacuum state is postulated in the case of the usual Fermi field. Each irreducible representation of equations (1.1) is determined by these requirements entirely. Thus we obtain the pure algebraic construction of the irreducible representations of equations (1.1). Of course, we have no need to consider the equivalent irreducible representations in this case.

To clarify the physical meaning of the minor vectors of the irreducible representations we write the operator of the particle number

$$N = \tfrac{1}{2}\int d\mathbf{x}[\psi^*(\mathbf{x})\,\psi(\mathbf{x}) - :\psi(\mathbf{x})\,\psi^*(\mathbf{x}):] \tag{2.12}$$

where we have used the notations for a normal product of operators

$$:\psi(\mathbf{x})\,\psi^*(\mathbf{y}): = \psi(\mathbf{x})\,\psi^*(\mathbf{y}) - \langle 0|\,\psi(\mathbf{x})\,\psi^*(\mathbf{y})\,|0\rangle \tag{2.13}$$

Due to equations (1.1) the operator N possesses all necessary properties

$$[\psi(\mathbf{x}), N] = \psi(\mathbf{x}), \qquad [\psi^*(\mathbf{x}), N] = -\psi^*(\mathbf{x}) \tag{2.14}$$

The action of operator N on the minor vectors gives

$$N|\mathbf{x}_1, \ldots, \mathbf{x}_q\rangle = q\,|\mathbf{x}_1, \ldots \mathbf{x}_q\rangle \tag{2.15}$$

Therefore the minor vectors correspond to q-particle states. There is a unique irreducible representation which includes the vacuum state. Other irreducible representations begin from one-, two-, etc., particle states.

We dwell on the two particular features of parastatistics in the general case in comparison with the usual statistics.

The first one is that the amplitudes of the general vector (2.10) are expressed by the linear combinations of its projections. For example, the two-particle amplitude of this vector in the Fock space $\mathscr{B}_0$ is given by

$$\begin{aligned}\Psi(\mathbf{x}_1, \mathbf{x}_2;) = \frac{s}{4(1-1/s)}\,&[\langle 0|\,\psi(\mathbf{x}_2)\,\psi(\mathbf{x}_1)\,|\Psi\rangle \\ &- (1-2/s)\langle 0|\,\psi(\mathbf{x}_1)\,\psi(\mathbf{x}_2)\,|\Psi\rangle]\end{aligned} \tag{2.16}$$

There is a simple correspondence between amplitudes and projections only in the case of the usual Fermi statistics and second-order para-Fermi statistics (Govorkov, 1969).

The second particular feature of parastatistics (including the case of the second order) takes into account the presence of several amplitudes, e.g. $\Psi(\mathbf{x}_1, \mathbf{x}_2;)$, $\Psi(\mathbf{x}_2, \mathbf{x}_1;)$, etc., which are differed by permutations of arguments.

The meaning of these amplitudes is hardly comprehended. However, we can attach a definite meaning to these amplitudes if we symmetrize them along the Young patterns. Then the symmetrized amplitudes determine the probability of finding the particles in their definite symmetrized states.

We can determine the representations of the paraparticle annihilation, $\psi(\mathbf{x})$, and creation, $\psi^*(\mathbf{x})$, operators on the columns of the symmetrized amplitudes like the Fock column. Then we could construct the total quantum mechanics of paraparticles and show in the non-relativistic case the connection between the first- and the second-quantized theories. It was carried out in the previous paper (Govorkov, 1969) for the case of pF_2-statistics. We cannot include here the explicit expressions for irreducible representations of parafield operators for the case of interest pF_3-statistics due to the lack of space.

3. *The Fock Type Columns of the Symmetrized Amplitudes*

We now present more details concerning the structure of the irreducible representations of equations (1.1) for the case of pF_3-statistics.

At first we determine the class of the many-particle functions on which pF_3-statistics is realized.

Due to equations (1.1) the amplitudes of the vector (2.10) have to satisfy the condition of permutations of their neighbouring three primary arguments

$$\Psi(\mathbf{x},\mathbf{y},\mathbf{z};\ldots) - \Psi(\mathbf{y},\mathbf{x},\mathbf{z};\ldots) - \Psi(\mathbf{z},\mathbf{x},\mathbf{y};\ldots) + \Psi(\mathbf{z},\mathbf{y},\mathbf{x};\ldots) = 0 \tag{3.1}$$

The symmetric combination in four primary arguments vanishes in accordance with equation (2.3)

$$\square\square\square\square \quad \Psi(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{u};\ldots) = 0 \tag{3.2}$$

Due to equation (2.7) the symmetric combination in three arguments vanishes in the space $\mathscr{B}_1$

$$\square\square\square \quad \Psi(\ldots\mathbf{x},\mathbf{y};\mathbf{z}) = 0 \tag{3.3}$$

Finally, it follows from equation (2.8) that the amplitudes are antisymmetrical with respect to their secondary arguments

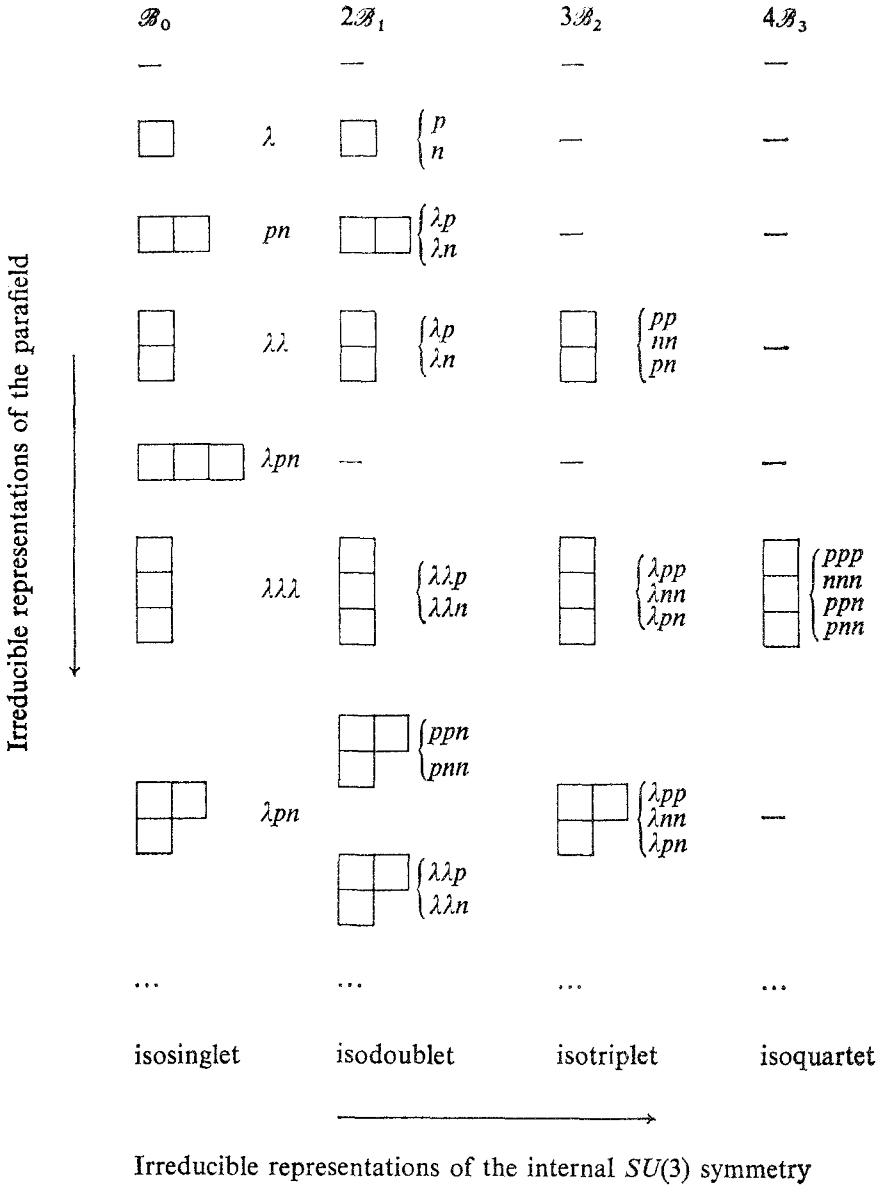
$$\Psi(\ldots;\ldots,\mathbf{x}_i,\mathbf{x}_{i+1},\ldots) = -\Psi(\ldots;\ldots,\mathbf{x}_{i+1},\mathbf{x}_i,\ldots) \tag{3.4}$$

We now limit ourselves to consideration of the states with no more than three particles. We then obtain the symmetrized amplitudes corresponding to the Young patterns indicated in Table 1.

The symbols of the irreducible representations are indicated in the upper line of Table 1. The number of the equivalent irreducible representations in the space of the Green-Ansatz are indicated before these symbols. Below these symbols the Fock-like columns of the symmetrized amplitudes give us the corresponding irreducible representations of equations (1.1). The gaps in the columns indicate the absence of the corresponding symmetrized functions.

There is only one Young pattern of a mixed symmetry for the three-particle states in the space $\mathscr{B}_0$ because of equation (3.1). There are two such

TABLE 1. Irreducible representations of the third order para-Fermi field and the 'quark' composition of the paraparticle states



patterns in the space $\mathscr{B}_1$ whereas equation (3.1) is invalid for the function with two primary and one secondary arguments. On the other hand, owing to equation (3.3), only two particles can be in the symmetric state in this irreducible representation.

Due to equation (3.4) the particles are not in the entirely symmetrical state in the space $\mathscr{B}_2$. For the same reason there is only one Young pattern of a mixed symmetry for the three-particle states in this irreducible representation.

Finally, there are only antisymmetrical three-particle states in the space $\mathscr{B}_3$ because of equation (3.4). Here we stop our consideration of irreducible representations because the next irreducible representation should begin from the four particle states.

The columns of Table 1 give us the irreducible representations of equations (1.1) for the case of interest pF_3-statistics. We now consider the states of the same number of paraparticles with the same Young pattern taken across different columns. We can interpret the states belonging to the different irreducible representations as the different internal states of paraparticles.

In the next section we establish the internal symmetry of such states and compare them with the states of ordinary fermions with three-valued internal degree of freedom.

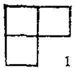
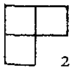
4. *Internal Symmetry of the Paraparticle States*

To perform the above-mentioned task it will be convenient to return to the decomposition (2.11) of the large space of the Green-Ansatz into the spaces of irreducible representations of equations (1.1).

We use the following classification for the paraparticle states with the same Young pattern. The states belonging to the essentially different irreducible representations are considered as the states with different internal quantum-number-like strangeness. The states from the equivalent irreducible representations are interpreted as the states forming 'isomultiplets' with the same 'strangeness'. Then our classification of multiplets coincides with that for $SU(3)$ symmetry. For example, we have for the three-particle states the following nomenclature.

The only symmetric state forms a singlet in $\mathscr{B}_0$.

The antisymmetric states form a decouplet, consisting of an isosinglet in $\mathscr{B}_0$, an isodoublet in $2\mathscr{B}_1$, an isotriplet in $3\mathscr{B}_2$, and an isoquartet in $4\mathscr{B}_3$.

The states of mixed symmetry form an octet, consisting of one isosinglet in $\mathscr{B}_0$, two isodoublets, corresponding to two independent Young patterns  1 and  2 in $2\mathscr{B}_1$, and one isotriplet in $3\mathscr{B}_2$.

We can now establish a one-to-one correspondence between the paraparticle states and the states of the ordinary fermions like quarks with internal degrees of freedom like isospin and strangeness. We introduce the notations λ, p and n for these fermions by analogy with the ones for usual quarks. The λ particle is a strange isosinglet, the non-strange particles p and n form an isodoublet. In Table 1 the corresponding 'quark' states are indicated on the right from the symmetrized paraparticle states which are presented by the corresponding Young patterns.

Now we return to the proper parafield theory. There is no reason to consider the equivalent irreducible representations of parafield in this theory. It is worthwhile to consider only essentially different irreducible representations. Then each state from the irreducible representations of the parafield corresponds to a whole isomultiplet of ordinary fermions. Thus the states from $\mathscr{B}_0$ correspond to isosinglets, the states from $\mathscr{B}_1$ correspond to whole isodoublets, etc. The theory of the para-Fermi field of order 3 is therefore equivalent to a theory with three types of ordinary fermions but two of them, non-strange fermions, are dynamically indistinguishable while the third, strange fermion, is dynamically distinguishable from non-strange fermions.

It turns out that the Fermi field operators corresponding to a strange fermion can be obtained in the framework of the proper parafield theory. Indeed, one can define the operators that satisfy the usual anticommutation relations for the Fermi field over the spaces of irreducible representations of the para-Fermi field. This Fermi field corresponds to the λ-fermions. Its creation operator is expressed in the form of an infinite set

$$\lambda^*(\mathbf{x}) = \frac{1}{\sqrt{3}}\psi^*(\mathbf{x}) + \int d\mathbf{y}\{a_1\psi^*(\mathbf{x})\psi^*(\mathbf{y})\psi(\mathbf{y}) + a_2\psi^*(\mathbf{y})\psi^*(\mathbf{x})\psi(\mathbf{y}) + a_3\psi^*(\mathbf{x})(:\psi(\mathbf{y})\psi^*(\mathbf{y}):) + a_4\psi^*(\mathbf{y})(:\psi(\mathbf{y})\psi^*(\mathbf{x}):)\} + \dots \tag{4.1}$$

where the coefficients assume the values

$$\left.\begin{aligned} a_1 &= \frac{1}{4} - \frac{1}{2\sqrt{3}} + \frac{1}{12\sqrt{5}}, \qquad & a_2 &= -\frac{1}{12} - \frac{1}{12\sqrt{5}} \\ a_3 &= -\frac{1}{4} + \frac{1}{2\sqrt{3}} - \frac{1}{4\sqrt{5}}, \qquad & a_4 &= -\frac{1}{4} + \frac{1}{4\sqrt{5}} \end{aligned}\right\} \tag{4.2}$$

In expression (4.1) we have cited only several first terms of the infinite set that correspond to the states with no more than two particles.

The annihilation operator of λ-fermions is hermitian conjugate to operator (4.1)

$$\lambda(\mathbf{x}) = (\lambda^*(\mathbf{x}))^* \tag{4.3}$$

(For a one-level system it was shown (Govorkov, 1971) that the Fermi operators can be defined by means of para-Fermi operators in the cases of odd-order parastatistics and it cannot be done in cases of even-order parastatistics.)

Now we can introduce the following operators whose eigenvalues determine a state of paraparticles.

Firstly we can introduce an operator similar to the baryonic charge. It is simply proportional to the particle number operator (2.12)

$$B = \tfrac{1}{3}N \tag{4.4}$$

Secondly we can define an operator similar to the hypercharge with the help of operator (4.1). But in accordance with (4.1) it is expressed in the form of the infinite set

$$\begin{aligned} Y &= B - \int d\mathbf{x}\lambda^*(\mathbf{x})\,\lambda(\mathbf{x}) \\ &= -\tfrac{1}{6}\int d\mathbf{x}\{\psi^*(\mathbf{x})\,\psi(\mathbf{x}) + (:\psi(\mathbf{x})\,\psi^*(\mathbf{x}):)\} \\ &\quad -\int d\mathbf{x}\int d\mathbf{y}\{\tfrac{1}{10}\psi^*(\mathbf{x})\,\psi^*(\mathbf{y})\,\psi(\mathbf{y})\,\psi(\mathbf{x}) \\ &\quad -\tfrac{1}{10}\psi^*(\mathbf{x})\,\psi^*(\mathbf{y})\,\psi(\mathbf{x})\,\psi(\mathbf{y}) + \tfrac{1}{6}\psi^*(\mathbf{x})(:\psi(\mathbf{y})\,\psi^*(\mathbf{y}):)\,\psi(\mathbf{x}) \\ &\quad -\tfrac{1}{5}\psi^*(\mathbf{x})(:\psi(\mathbf{x})\,\psi^*(\mathbf{y}):)\,\psi(\mathbf{y})\} + \ldots \end{aligned} \tag{4.5}$$

Thus any state in the irreducible representations of parafield $\mathscr{B}_0$, $\mathscr{B}_1$, $\mathscr{B}_2$, etc. can be labelled by the eigenvalue of the 'baryonic' charge B, by the type of the Young pattern and by the eigenvalue of the 'hypercharge' Y. But we cannot introduce the 'isospin' operators in the parafield theory in accordance with the foregoing indistinguishability of the equivalent irreducible representations of parafield.

Now we consider possible interactions of parafield.

Hitherto we considered the non-relativistic Schrödinger spinor field. It is necessary to proceed to the relativistic Dirac field for the consideration of parafield interactions. The transition consists in a few formal alterations. Now the field $\psi(x)$ is the Dirac field depending on the four-dimensional, space and time, variables x. For the free field the operator $\psi(x)$ contains the annihilation operators of paraparticles and the creation operators of anti-paraparticles. On the contrary, the Dirac-conjugate operator $\bar{\psi}(x)$ contains the creation operators of paraparticles and the annihilation operators of anti-paraparticles. The Dirac three-dimensional δ-function should be changed by the well-known covariant S-function for the Dirac field in the Green commutation relations (1.1) and in other expressions. The integration over the volume should be changed by the integration over the space-like hypersurface. Finally, the normal product should be taken for all products of the field operators. As usual, the normal product of the field operators is defined as a product with the subtraction of its vacuum expectation.

We shall demand a locality from parafield interactions and from currents entering them. It means that the equal-time commutator of the considered quantities of the currents, for example, should be equal to zero

$$[j(x), j(y)]_{x_0=y_0} = 0 \tag{4.6}$$

We shall call the interaction 'strong' if it includes the 'baryonic' current in the form of the commutator of a parafield

$$j_\mu{}^B(x) = \tfrac{1}{6}[:\bar{\psi}(x)\,\gamma_\mu\,\psi(x): - :\psi(x)\,\gamma_\mu\,\bar{\psi}(x):] \tag{4.7}$$

The locality of this current is easily proved either by means of the Green-Ansatz (Greenberg & Messiah, 1965) or by direct application of the Green algebra when commutator (4.7) acts on the states of the irreducible representations of parafield. The inclusion of the 'strong' interaction may lead to different unitary multiplets being separated from one another, but it does not split the states belonging to the same unitary multiplet.

However, one can construct the other local current by means of the Fermi operators (4.1 and 4.3) taken in their relativistic forms

$$j_\mu{}^\lambda(x) = :\bar{\lambda}(x)\gamma_\mu\lambda(x): \tag{4.8}$$

We shall call the interaction which includes this current 'medium-strong'. It breaks the internal unitary symmetry of paraparticle states. Indeed, the 'strange' fermion λ is dynamically distinguished by this interaction with respect to 'non-strange' fermions p and n.

It must be emphasised that the 'strong' and 'medium-strong' interactions are formulated in the framework of the proper parafield theory. They work inside each irreducible representation of parafield.

If we want to consider the 'electromagnetic' violation of the internal symmetry of paraparticle states, then we should introduce an interaction which breaks the indistinguishability of the equivalent irreducible representations of parafield. Further, if we want to consider the transitions between the internal states of paraparticles owing to the 'weak' interaction, then we should introduce an interaction which works between the irreducible representations of parafield. Both of them cannot be introduced in the framework of the proper parafield theory.

These latter interactions can be introduced by the consideration of a parafield theory in the large space of the Green-Ansatz. However, the latter theory is virtually the theory of three different parafields. Indeed, we can determine by means of the Green-components satisfying (2.1) two other linear combinations besides the usual sum

$$\left.\begin{aligned} \psi(x) &= \psi^{(1)}(x) + \psi^{(2)}(x) + \psi^{(3)}(x) \\ g'(x) &= k\psi^{(1)}(x) + \bar{k}\psi^{(2)}(x) + \psi^{(3)}(x) \\ g''(x) &= \bar{k}\psi^{(1)}(x) + k\psi^{(2)}(x) + \psi^{(3)}(x) \end{aligned}\right\} \tag{4.9}$$

where $k = \exp(2\pi i/3)$, $\bar{k}$ is complex conjugate to k. The two latter auxiliary combinations in (4.9), as well as the former sum in (4.9), satisfy equation (1.1) and are the parafields too.

Thus we conclude that the theory of proper para-Fermi field of order 3 is equivalent to the theory of $SU(3)$ symmetry of three kinds of fermion, like quarks, with internal degrees of freedom like isospin and strangeness. But each parafield state corresponds to the whole isomultiplet of these fermions and one can introduce the 'strong' and 'medium-strong' interactions but cannot introduce the 'electromagnetic' and 'weak' interactions in the framework of the parafield theory.

We add that one can constitute an interaction which breaks the law of conservation of paraparticles with the help of the Fermi field (4,1) and other usual Fermi fields. It is curious that this interaction could break only the law of strange particle number conservation. Thus the law of the 'conservation of statistics' obtained by Kamefuchi & Strathdee (1963) and Greenberg & Messiah (1965) has, in our opinion, a restricted meaning.

5. *Conclusion*

We have considered an ensemble of all separable irreducible representations of para-Fermi field of order 3 in the configuration space.

It was shown that these symmetrized along the Young pattern state, belonging to the irreducible representations of the parafield, can be labelled by the eigenvalues of the operators that are similar to baryonic charge and hypercharge.

It turns out that one Fermi field operator can be defined in the framework of the parafield theory.

We arrived at the conclusion that such a theory is equivalent to the limited theory of $SU(3)$ symmetry of three kinds of usual fermions, like quarks, with internal degree of freedom like isospin and strangeness. The limitations of this comparison lies in the fact that, firstly, each parafield state corresponds to whole isomultiplet of the fermions, and, secondly, only the 'strange' fermion can be dynamically distinguished from other 'nonstrange' fermions in the framework of the parafield theory.

Thus, the considered theory appears as some model of the theory of $SU(3)$ symmetry of physical elementary particles whose strong and medium-strong interactions which conserve baryonic and hyper-charges are switched on but the electromagnetic and weak interactions are switched off. For switching on these latter it is necessary to consider distinguishability of equivalent irreducible representations and transitions between irreducible representations of the parafield. It is impossible to do in the framework of the proper parafield theory.

It follows from the previous consideration (Govorkov, 1969) of a theory of the second order para-Fermi field that this latter corresponds to some model of the theory of $SU(2)$(isospin) symmetry of physical particles like protons and neutrons whose strong and electromagnetic interactions are switched on but the weak interaction is switched off.

The connection between parafield theory and a theory of usual fields with internal degrees of freedom was considered by many authors (Green, 1953; Chernikov, 1962; Landshoff & Stapp, 1967; Carpenter, 1970; Drühl *et al.*, 1970). But Landshoff & Stapp (1967) and Drühl *et al.* (1970) asserted that the para-Fermi field theory is equivalent to a theory of several types of ordinary fermions that are dynamically indistinguishable. It follows from the results of the present paper as well as from the previous paper (Govorkov 1969) that the latter assertion is not true at least for the cases of para-Fermi fields of order 2 and 3. In these cases we composed local interactions which distinguished one of these kinds of ordinary fermions from the others.

Further, Kamefuchi & Strathdee (1963) and Greenberg & Messiah (1965) concluded that the requirement of locality of a parafield interaction means the 'conservation of statistics'. We showed that it was not true for the para-Fermi field of order 3, if we composed the interaction of an ordinary Fermi field with the Fermi field which can be defined in the framework of the para-Fermi field theory. It is curious that the conservation of the number of only 'strange' fermions could break down.

There are many papers in literature (Kamefuchi & Takahashi, 1962; Galindo & Indurain, 1963; Greenberg, 1966; Landshoff & Stapp, 1967; Yamada, 1968; Carpenter, 1970; Ohnuki & Kamefuchi, 1970; Stolt & Taylor, 1970) devoted to the investigation of the connection between the first- and second-quantized theories for parafield.

The reduction of the para-Fermi statistics to the ordinary Fermi statistics allows us to simplify this problem. Thus, for example, many authors emphasized that there is a distinction between the 'particle permutations' and the 'place permutations' in the theory of paraparticles (Landshoff & Stapp, 1967; Yamada, 1968; Ohnuki & Kamefuchi, 1970; Stolt & Taylor, 1970). Now it is clear that the 'particle permutations' imply the permutations of the corresponding ordinary fermions whereas the 'space permutations' correspond to the permutations of the fermion states without permutations of their internal states. Under the latter permutations the identical fermions could find themselves in the same state and the wave function vanishes. The latter led to the paradox of the parafield (Galindo & Indurain, 1963).

Finally, there was an alluring attempt to apply the para-Fermi statistics of order 3 to the physical quarks in the connection with the problem of the symmetry of their ground state in baryons in the quark model of hadrons (Greenberg, 1964). Now we can see that this suggestion is equivalent to the assumption of a new $SU(3)'$ symmetry for the quarks with the above-mentioned limitations of its violations. Thus, for example, if we consider only the Fock representation of a parafield (the first column of the Table 1) then it corresponds to the consideration of only the isosinglets of the new $SU(3)'$ symmetry. If we consider only symmetrical three-particle state, as proposed for baryons (Greenberg, 1964), then it corresponds to the consideration of only unitary singlet of the new $SU(3)'$ symmetry. Thus we have no new matter as paraquarks but we have the old quarks with a new limited internal $SU(3)'$ symmetry.

In conclusion we remark that there is another attempt to relate the parafields to the internal symmetries of elementary particles which is based on the matrix representation of fields (Scharfstein, 1968, 1969).

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Appendix

In the present Appendix we prove the properties (2.6)–(2.8) of minor vectors in the framework of the Green-Ansatz.

It can be proved that the requirement (2.5) implies the following restrictions on the coefficients of minor vectors

$$\left.\begin{aligned}&\sum_{A_1=1}^{s} y_{A_1A_2\ldots A_q}=0\\&\sum_{A_2=1}^{s} \varepsilon_{A_1A_2}y_{A_1A_2\ldots A_q}=0\\&\ldots\\&\sum_{A_q=1}^{s} \varepsilon_{A_1A_q}\varepsilon_{A_2A_q}\ldots\varepsilon_{A_{q-1}A_q}y_{A_1A_2\ldots A_q}=0\end{aligned}\right\}\tag{A.1}$$

where we designated $\varepsilon_{AB}=2\delta_{AB}-1$.

Firstly, we prove equation (2.6). Using (2.1), (2.2) and (2.4) we have

$$\begin{aligned}&\psi(\mathbf{x})\,\psi^*(\mathbf{y})\,|\mathbf{x}_1,\ldots,\mathbf{x}_q\rangle\\&=\sum_{\substack{A,B,\\A_1,\ldots,A_q=1}}^{s} y_{A_1\ldots A_q}\psi^{A}(\mathbf{x})\,\psi^{B*}(\mathbf{y})\,\psi^{A_1*}(\mathbf{x}_1)\ldots\psi^{A_q*}(\mathbf{x}_q)\,|0\rangle\\&=s\,\delta^{(3)}(\mathbf{x}-\mathbf{y})\,|\mathbf{x}_1,\ldots,\mathbf{x}_q\rangle\\&\quad+\sum_{i=1}^{q}\delta^{(3)}(\mathbf{x}-\mathbf{x}_i)\Bigg\{(-1)^i\sum_{\substack{A,B,\\A_1,\ldots,A_q=1}}^{s}\varepsilon_{AB}\,\varepsilon_{AA_1}\ldots\varepsilon_{AA_{i-1}}\\&\quad\times\delta_{AA_i}y_{A_1\ldots A_q}\psi^{B*}(\mathbf{y})\,\psi^{A_1*}(\mathbf{x}_1)\ldots\psi^{A_{i-1}*}(\mathbf{x}_{i-1})\\&\quad\times\psi^{A_{i+1}*}(\mathbf{x}_{i+1})\ldots\psi^{A_q*}(\mathbf{x}_q)\,|0\rangle\Bigg\}\end{aligned}\tag{A.2}$$

Let us consider the expression in the braces in the right-hand side of (A.2) for fixed i. We write $\varepsilon_{AB}=2\delta_{AB}-1$ and then sum up over A and B. We arrived at

$$\begin{aligned}&2(-1)^i\sum_{A_1,\ldots,A_q=1}^{s}\varepsilon_{A_1A_i}\ldots\varepsilon_{A_{i-1}A_i}y_{A_1\ldots A_q}\psi^{A_i*}(\mathbf{y})\\&\times\psi^{A_1*}(\mathbf{x}_1)\ldots\psi^{A_{i-1}*}(\mathbf{x}_{i-1})\,\psi^{A_{i+1}*}(\mathbf{x}_{i+1})\ldots\psi^{A_q*}(\mathbf{x}_q)\,|0\rangle\\&-(-1)^i\sum_{\substack{B,\\A_1,\ldots,A_{i-1},A_{i+1},\ldots,A_q=1}}^{s}\left[\sum_{A_i=1}^{s}\varepsilon_{A_1A_i}\ldots\varepsilon_{A_{i-1}A_i}y_{A_1\ldots A_q}\right]\\&\times\psi^{B*}(\mathbf{y})\,\psi^{A_1*}(\mathbf{x}_1)\ldots\psi^{A_{i-1}*}(\mathbf{x}_{i-1})\,\psi^{A_{i+1}*}(\mathbf{x}_{i+1})\ldots\psi^{A_q*}(\mathbf{x}_q)\,|0\rangle\end{aligned}\tag{A.3}$$

When the operator $\psi^{A_i *}(\mathbf{y})$ is moved to its own place the former sum from (A.3) gives us

$$-2|\mathbf{x}_1,\ldots,\mathbf{x}_{i-1},\mathbf{y},\mathbf{x}_{i+1},\ldots,\mathbf{x}_q\rangle \tag{A.4}$$

The latter sum in (A.3) vanishes because the expression in the square brackets is equal to zero due to the ith equation of (A.1). Equations (A.2) and (A.4) result in (2.6) that is sought for.

Now we prove equation (2.7). Using the definition of the symmetric combination and equations (2.2) and (2.4) we write

$$\boxed{}\boxed{}\boxed{\ldots}\boxed{}\quad \psi^*(\mathbf{x}_1)\ldots\psi^*(\mathbf{x}_{s-1})|\mathbf{x}_s\rangle$$
$$= \sum_{A_1,\ldots,A_s=1}^{s} y_{A_s}\Big\{\sum_{\mathscr{P}} \psi^{A_1 *}(\mathbf{x}_{\mathscr{P}_1})\ldots\psi^{A_s *}(\mathbf{x}_{\mathscr{P}_s})|0\rangle\Big\} \tag{A.5}$$

where the sum in the braces is taken over all $s!$ permutations, $\mathscr{P}$, of the indices $1,\ldots,s$. In fact only the terms with different indices $A_1 \neq A_2 \neq \ldots \neq A_s$ give the contributions to the sum because the terms with equal at least two indices vanish due to equations (2.1). Thus one can write the right-hand side of equation (A.5) as

$$\Big(\sum_{A_s=1}^{s} y_{A_s}\Big)\Big\{\sum_{\mathscr{P}} \psi^{(1)*}(\mathbf{x}_{\mathscr{P}_1})\ldots\psi^{(s)*}(\mathbf{x}_{\mathscr{P}_s})\Big\}|0\rangle \tag{A.6}$$

The first multiplier is equal to zero because of the first equation (A.1). Thus equation (2.7) has been proved.

It remains to prove equation (2.8) for the particular cases of $s = 2$ and $s = 3$.

Let us consider any couple of neighbouring equations, for example, $(i-1)$th and ith, from equations (A.1). We pick out the terms with the identical indices $A_i = A_{i-1}$ and designate the indices A_i and A_{i-1} by A and B, respectively, in the first equation and, on the contrary, by B and A in the second equation. It yields

$$\left.\begin{aligned}
&\varepsilon_{A_1 A}\ldots\varepsilon_{A_{i-2}A}\, y_{A_1\ldots AA\ldots A_q} \\
&\quad + \sum_{\substack{B=1\\(B\neq A)}}^{s} \varepsilon_{A_1 B}\ldots\varepsilon_{A_{i-2}B}\, y_{A_1\ldots BA\ldots A_q} = 0 \\
&- \varepsilon_{A_1 A}\ldots\varepsilon_{A_{i-2}A}\, y_{A_1\ldots AA\ldots A_q} \\
&\quad + \sum_{\substack{B=1\\(B\neq A)}}^{s} \varepsilon_{A_1 B}\ldots\varepsilon_{A_{i-2}B}\, y_{A_1\ldots AB\ldots A_q} = 0.
\end{aligned}\right\} \tag{A.7}$$

Under summation it leads to

$$\sum_{\substack{B=1\\(B\neq A)}}^{s} \varepsilon_{A_1 B}\ldots\varepsilon_{A_{i-2}B}\, c_{AB} = 0, \qquad A = 1,2,\ldots,s \tag{A.8}$$

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